Parametricity versus the Universal Type – Technical Appendix

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Abstract

This technical appendix contains all the proofs of the paper.

Contents

1 Proof of Theorem 2.4

To make this proof, we need a good formulation of parametricity for $\lambda^{\mathbf{F}}$. This is in itself non-trivial and so, rather than build this from scratch ourselves, we use Dreyer et al. [2009]'s logical relations for System F. Note that we do not use the journal version of that paper [Dreyer et al., 2011a], because that LR is built a bit differently in order to guarantee completeness of the LR rather than just soundness, but this alternative setup seems to make it impossible to prove our Bind2-Schizo rule (see below). We do take care to avoid the rules from Dreyer et al. [2009] that were flagged as incorrect in Dreyer et al. [2011a].

Conveniently, the language F^{μ} that Dreyer et al. consider is the same as ours except that they additionally have natural numbers.

Our proof of contextual equivalence for $\mathbf{t_u}$ and $\mathbf{t_d}$ relies on the idea of using a term at two different instantiations of its type. For this, we require a special version of rule 6 from Dreyer et al.'s Figure 5. We calll this a schizophrenic bind because it interprets the same term in two different semantic type environments. We will need this to do what we described above: exploit the polymorphism of $x' : \forall\alpha.\, (\alpha \to \beta) \times (\beta \to \alpha)$ in α twice at two different interpretations of α.

Lemma 1.1 (Bind-Schizo is derivable.). The following rule Bind-Schizo is derivable.

$$\frac{\begin{array}{cc} \Gamma; \Delta; \Theta \vdash (e_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho & \Gamma; \Delta; \Theta \vdash (e_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho' \\ \rho_1 = \rho'_1 & \rho_2 = \rho'_2 \\ \multicolumn{2}{c}{\Theta' = \Theta, (x_1, x_2) \in \mathcal{V}[\![\tau]\!]\rho, (x_1, x_2) \in \mathcal{V}[\![\tau]\!]\rho', e_1 \rightsquigarrow^* x_1, e_2 \rightsquigarrow^* x_2} \\ \multicolumn{2}{c}{\Gamma, x_1, x_2; \Delta; \Theta' \vdash (E[x_1], f) \in \mathcal{E}[\![\tau']\!]\rho''} \end{array}}{\Gamma; \Delta; \Theta \vdash (E[e_1], f) \in \mathcal{E}[\![\tau']\!]\rho''}$$

Proof. For Rule Bind-Schizo, we mimic Dreyer et al.'s proof of Rule 6 [Dreyer et al., 2011b, Appendix C].

Define $P(t_u)$ to be the proposition

$$\begin{aligned} &\forall x_1, x_2.\, ((x_1, x_2) \in \mathcal{V}[\![\tau]\!]\rho \wedge (x_1, x_2) \in \mathcal{V}[\![\tau]\!]\rho' \wedge \\ &\qquad\qquad t_u \rightsquigarrow^* x_1 \wedge e_2 \rightsquigarrow^* x_2) \Rightarrow (E[x_1], f) \in \mathcal{E}[\![\tau']\!]\rho'' \end{aligned}$$

We want to prove that

$$\forall t_u.\, ((t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho \wedge (t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho' \wedge P(t_u)) \Rightarrow (E[t_u], f) \in \mathcal{E}[\![\tau']\!]\rho''$$

By Löb-induction, we assume this proposition is true later and proceed to prove it now. So assume that $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho$, $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho'$ and $P(t_u)$ and we want to prove $(E[t_u], f) \in \mathcal{E}[\![\tau']\!]\rho''$.

First, suppose that $E[t_u] \rightsquigarrow^0 x_1$ for some x_1. Then, it must be the case that $t_u \rightsquigarrow^0 y_1$ for some y_1 and also that $E[t_u] \rightsquigarrow^0 E[y_1] \rightsquigarrow^0 x_1$. Since $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho$, we know there exists some y_2 such that $e_2 \rightsquigarrow^* y_2$ and $(y_1, y_2) \in \mathcal{V}[\![\tau]\!]\rho$. Similarly, from $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho'$, we know there exists some y'_2 such that $e_2 \rightsquigarrow^* y'_2$ and $(y_1, y'_2) \in \mathcal{V}[\![\tau]\!]\rho'$. Since y_2 and y'_2 are both values, a standard determinacy lemma implies that $y_2 = y_2'$. By $P(t_u)$, we then know that that $(E[x_1], f) \in \mathcal{E}[\![\tau']\!]\rho''$.

Second, suppose that $E[t_u] \rightsquigarrow^1 t_u'$. There are two cases:

Case 1 There exists y_1 such that $t_u \rightsquigarrow^0 y_1$ and also that $E[t_u] \rightsquigarrow^0 E[y_1] \rightsquigarrow^1 t_u'$. The proof is identical to the previous case shown above.

Case 2 There exists u_1 such that $t_u \rightsquigarrow^1 u_1$ and also that $E[t_u] \rightsquigarrow^1 E[y_1] \rightsquigarrow^0 t_u'$. Since $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho$, we know that $\triangleright(u_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho$ (by the definition of the expression relation in Dreyer et al.'s Figure 4). Similarly, from $(t_u, e_2) \in \mathcal{E}[\![\tau]\!]\rho'$, we know that $\triangleright(u_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho'$. It is also easy to show that $P(t_u)$ implies $P(u_1)$. Thus, by appealing to our Löb-inductive hypothesis, we have that $\triangleright(E[u_1], f) \in \mathcal{E}[\![\tau']\!]\rho''$. Then, by rule 5 from Dreyer et al.'s Figure 5, we have that $(E[t_u], f) \in \mathcal{E}[\![\tau']\!]\rho''$.

Lemma 1.2 (Bind2-Schizo is derivable.). The following rule Bind2-Schizo is derivable.

$$\frac{\begin{array}{cc} \Gamma; \Delta; \Theta \vdash (e_1, e_2) \in \mathcal{E}^{\approx}[\![\tau]\!]\rho & \Gamma; \Delta; \Theta \vdash (e_1, e_2) \in \mathcal{E}^{\approx}[\![\tau]\!]\rho' \\ \rho_1 = \rho_1' & \rho_2 = \rho_2' \\ \multicolumn{2}{c}{\Theta' = \Theta, (x_1, x_2) \in \mathcal{V}^{\approx}[\![\tau]\!]\rho, (x_1, x_2) \in \mathcal{V}^{\approx}[\![\tau]\!]\rho', e_1 \rightsquigarrow^* x_1, e_2 \rightsquigarrow^* x_2} \\ \multicolumn{2}{c}{\Gamma, x_1, x_2; \Delta; \Theta' \vdash (E_1[x_1], E_2[x_2]) \in \mathcal{E}^{\approx}[\![\tau']\!]\rho''} \end{array}}{\Gamma; \Delta; \Theta \vdash (E_1[e_1], E_2[e_2]) \in \mathcal{E}^{\approx}[\![\tau']\!]\rho''}$$

Proof. Follows from the definition of the symmetric logical relation and Lemma 1.1's rule Bind-Schizo, together with the fact that $e_2 \rightsquigarrow^* x_2$ implies that $E[e_2] \rightsquigarrow^* E[x_2]$.

Lemma 1.3 (Terms are equivalent). $\mathbf{t_u} \simeq_{ctx} \mathbf{t_d}$

Proof. For brevity, we will omit any proofs about well-typedness of terms or the fact that a term is a value, as these can be easily filled in as needed.

For notation purposes: when we mention a proof obligation that is a judgement inside the LSLR logic, this means that the judgement has to be provable inside LSLR.

Note that we use the symmetric version of Dreyer et al.'s logical relation, discussed in Dreyer et al. [2009, pp.6–7].

1. Suffices: $\emptyset \vdash \mathbf{t_u} \approx^{\mathbf{log}} \mathbf{t_d}$

   Proof: By Dreyer et al.'s Theorem 4.4 in two directions.
2. Suffices:

$$\begin{array}{l} x : \mathrm{Univ}, \beta, x' : \forall\alpha. (\alpha \to \beta) \times (\beta \to \alpha) \vdash \\ \mathrm{let}\ x'' : (\mathtt{Unit} \to \beta) \times (\beta \to \mathtt{Unit}) = x'\ \mathtt{Unit}\ \mathrm{in}\ x''.2\ (x''.1\ \mathtt{unit}) \approx^{\mathrm{log}} \\ \mathrm{let}\ x'' : (\mathtt{Unit} \to \beta) \times (\beta \to \mathtt{Unit}) = x'\ \mathtt{Unit}\ \mathrm{in}\ (x''.2\ (x''.1\ \mathtt{unit}); \omega_{\mathtt{Unit}}) \\ \ : \mathtt{Unit} \end{array}$$

   Proof: By a number of compatibility lemmas, as one would expect to be used in the proof of Dreyer et al.'s Theorem 4.3.

3. SUFFICES: Define

$$\mathbf{\Gamma} \stackrel{\mathsf{def}}{=} \beta_\mathbf{1}, \beta_\mathbf{2}, \mathbf{x}'_\mathbf{1} : \forall\alpha.\, (\alpha \to \beta_\mathbf{1}) \times (\beta_\mathbf{1} \to \alpha), \mathbf{x}'_\mathbf{2} : \forall\alpha.\, (\alpha \to \beta_\mathbf{2}) \times (\beta_\mathbf{2} \to \alpha)$$

$$\mathbf{\Delta} \stackrel{\mathsf{def}}{=} \mathbf{r}_\beta : \mathrm{VRel}(\beta_\mathbf{1}, \beta_\mathbf{2})$$

$$\rho \stackrel{\mathsf{def}}{=} \{\beta \mapsto (\beta_\mathbf{1}, \beta_\mathbf{2}, \mathbf{r}_\beta)\}$$

$$\mathbf{\Theta} \stackrel{\mathsf{def}}{=} (\mathbf{x}'_\mathbf{1}, \mathbf{x}'_\mathbf{2}) \in \mathcal{V}[\![\forall\alpha.\, (\alpha \to \beta) \times (\beta_\mathbf{2} \to \alpha)]\!]\rho$$

Then

$$\Gamma; \Delta; \Theta \vdash$$
$$(\text{let } x'' : (\texttt{Unit} \to \beta_1) \times (\beta_1 \to \texttt{Unit}) = x'_1\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit}),$$
$$\text{let } x'' : (\texttt{Unit} \to \beta_2) \times (\beta_2 \to \texttt{Unit}) = x'_2\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit}); \omega_{\texttt{Unit}})$$
$$\in \mathcal{E}^{\approx}[\![\texttt{Unit}]\!]\rho$$

PROOF: By definition of the symmetric logical equivalence relation and a few weakenings (i.e. dropping assumptions) in the LSLR logic.

4. Define

- $R_\alpha \stackrel{\mathsf{def}}{=} (\texttt{Unit}, \texttt{Unit}, (\mathbf{x_1} : \texttt{Unit}, \mathbf{x_2} : \texttt{Unit}).\, \mathrm{False})$
- $R'_\alpha \stackrel{\mathsf{def}}{=} (\texttt{Unit}, \texttt{Unit}, (\mathbf{x_1} : \texttt{Unit}, \mathbf{x_2} : \texttt{Unit}).\, \mathrm{True})$.

Then both R_α and R'_α are valid relations.

5.
$$\Gamma; \Delta; \Theta \vdash$$
$$(x'_1\ [\texttt{Unit}], x'_2\ [\texttt{Unit}]) \in \mathcal{E}[\![(\alpha \to \beta) \times (\beta \to \alpha)]\!]\rho, \alpha \mapsto R_\alpha$$

and

$$\Gamma; \Delta; \Theta \vdash$$
$$(x'_1\ [\texttt{Unit}], x'_2\ [\texttt{Unit}]) \in \mathcal{E}[\![(\alpha \to \beta) \times (\beta \to \alpha)]\!]\rho, \alpha \mapsto R'_\alpha$$

PROOF: By definition of $\mathcal{V}[\![\forall\alpha.\, \tau]\!]\rho$ in Dreyer et al.'s Figure 4, using 4.

6. SUFFICES: Define

$$\mathbf{\Gamma}' \stackrel{\mathsf{def}}{=} \mathbf{\Gamma}, \mathbf{x}''_\mathbf{1}, \mathbf{x}''_\mathbf{2}$$

$$\mathbf{\Theta}' \stackrel{\mathsf{def}}{=} \mathbf{\Theta}, (\mathbf{x}''_\mathbf{1}, \mathbf{x}''_\mathbf{2}) \in \mathcal{V}^{\approx}[\![(\alpha \to \beta) \times (\beta \to \alpha)]\!](\rho, \alpha \mapsto \mathbf{R}_\alpha),$$
$$(\mathbf{x}''_\mathbf{1}, \mathbf{x}''_\mathbf{2}) \in \mathcal{V}^{\approx}[\![(\alpha \to \beta) \times (\beta \to \alpha)]\!](\rho, \alpha \mapsto \mathbf{R}'_\alpha)$$

Then

$$\Gamma'; \Delta; \Theta' \vdash$$
$$(\text{let } x'' : (\texttt{Unit} \to \beta_1) \times (\beta_1 \to \texttt{Unit}) = x''_1 \text{ in } x''.2\ (x''.1\ \texttt{unit}),$$
$$\text{let } x'' : (\texttt{Unit} \to \beta_2) \times (\beta_2 \to \texttt{Unit}) = x''_2 \text{ in } x''.2\ (x''.1\ \texttt{unit}); \omega_{\texttt{Unit}})$$
$$\in \mathcal{E}^{\approx}[\![\texttt{Unit}]\!]\rho$$

PROOF: By Lemma 1.2 (BIND2-SCHIZO is derivable) and rule BIND2-SCHIZO, with 5.

7. SUFFICES: Then

$$\mathbf{\Gamma}'; \mathbf{\Delta}; \mathbf{\Theta}' \vdash (\mathbf{x}''_\mathbf{1}.\mathbf{2}\ (\mathbf{x}''_\mathbf{1}.\mathbf{1}\ \texttt{unit}), \mathbf{x}''_\mathbf{2}.\mathbf{2}\ (\mathbf{x}''_\mathbf{2}.\mathbf{1}\ \texttt{unit}); \omega_{\texttt{Unit}}) \in \mathcal{E}^{\approx}[\![\texttt{Unit}]\!]\rho$$

PROOF: By 6, using Rules 3 and 4 from Dreyer et al.'s Figure 5 and the obvious evaluations.

8. We have that

$$\mathbf{\Gamma}'; \mathbf{\Delta}; \mathbf{\Theta}' \vdash (\texttt{unit}, \texttt{unit}) \in \mathcal{V}^{\approx}[\![\mathbf{X}]\!](\rho, \alpha \mapsto \mathbf{R}'_\alpha)$$
$$\mathbf{\Gamma}'; \mathbf{\Delta}; \mathbf{\Theta}' \vdash (\mathbf{x}''_\mathbf{1}.\mathbf{1}, \mathbf{x}''_\mathbf{2}.\mathbf{1}) \in \mathcal{E}^{\approx}[\![(\mathbf{X} \to \beta)]\!](\rho, \alpha \mapsto \mathbf{R}'_\alpha)$$

$$\mathbf{\Gamma}'; \mathbf{\Delta}; \mathbf{\Theta}' \vdash (\mathbf{x_1''.1}\ \texttt{unit}, \mathbf{x_2''.1}\ \texttt{unit}) \in \mathcal{E}^{\approx}[\![\beta]\!](\rho, \alpha \mapsto \mathbf{R}'_\alpha)$$
$$\mathbf{\Gamma}'; \mathbf{\Delta}; \mathbf{\Theta}' \vdash (\mathbf{x_1''.1}\ \texttt{unit}, \mathbf{x_2''.1}\ \texttt{unit}) \in \mathcal{E}^{\approx}[\![\beta]\!]\rho$$

PROOF: By 6, a number of applications of the symmetric analogues of rules 1, 7, a compatibility lemma for the first projection, the definition of $\mathcal{V}^{\approx}[\![\alpha]\!]\rho$ and Lemma 4.1 from Dreyer et al.

9. Define

$$\mathbf{\Gamma}'' \overset{\mathsf{def}}{=} \mathbf{\Gamma}', \mathbf{x_1'''}, \mathbf{x_2'''}$$
$$\mathbf{\Theta}'' \overset{\mathsf{def}}{=} \mathbf{\Theta}', (\mathbf{x_1'''}, \mathbf{x_2'''}) \in \mathcal{V}^{\approx}[\![\beta]\!]\rho, (\mathbf{x_1''.1}\ \texttt{unit}) \rightsquigarrow^* \mathbf{x_1'''}, (\mathbf{x_2''.1}\ \texttt{unit}) \rightsquigarrow^* \mathbf{x_2'''}$$

SUFFICES: $\mathbf{\Gamma}''; \mathbf{\Delta}; \mathbf{\Theta}'' \vdash (\mathbf{x_1''.2}\ \mathbf{x_1'''}, \mathbf{x_2''.2}\ \mathbf{x_2'''}; \omega_{\texttt{Unit}}) \in \mathcal{E}^{\approx}[\![\texttt{Unit}]\!]\rho$

PROOF: By Rule 6S from Dreyer et al.'s Figure 6, using 8.

10. We have that

$$\mathbf{\Gamma}''; \mathbf{\Delta}; \mathbf{\Theta}'' \vdash (\mathbf{x_1'''}, \mathbf{x_2'''}) \in \mathcal{V}^{\approx}[\![\beta]\!]\rho$$
$$\mathbf{\Gamma}''; \mathbf{\Delta}; \mathbf{\Theta}'' \vdash (\mathbf{x_1''.2}, \mathbf{x_2''.2}) \in \mathcal{E}^{\approx}[\![(\beta \to \alpha)]\!](\rho, \alpha \mapsto \mathbf{R}_\alpha)$$
$$\mathbf{\Gamma}''; \mathbf{\Delta}; \mathbf{\Theta}'' \vdash (\mathbf{x_1''.2}\ \mathbf{x_1'''}, \mathbf{x_2''.2}\ \mathbf{x_2'''}) \in \mathcal{E}^{\approx}[\![\alpha]\!](\rho, \alpha \mapsto \mathbf{R}_\alpha)$$

PROOF: By 6, a number of applications of the symmetric analogues of rules 1, 7, a compatibility lemma for the second projection, the definition of $\mathcal{V}^{\approx}[\![\beta]\!]\rho$ and Lemma 4.1 from Dreyer et al.

11. Define

$$\mathbf{\Gamma}''' \overset{\mathsf{def}}{=} \mathbf{\Gamma}'', \mathbf{x_1''''}, \mathbf{x_2''''}$$
$$\mathbf{\Theta}''' \overset{\mathsf{def}}{=} \mathbf{\Theta}'', (\mathbf{x_1''''}, \mathbf{x_2''''}) \in \mathcal{V}^{\approx}[\![\alpha]\!](\rho, \alpha \mapsto \mathbf{R}_\alpha), (\mathbf{x_1''.2}\ \mathbf{x_1'''}) \rightsquigarrow^* \mathbf{x_1''''}, (\mathbf{x_2''.2}\ \mathbf{x_2'''}) \rightsquigarrow^* \mathbf{x_2''''}$$

SUFFICES: $\mathbf{\Gamma}'''; \mathbf{\Delta}; \mathbf{\Theta}''' \vdash (\mathbf{x_1''''}, (\mathbf{x_2''''}; \omega_{\texttt{Unit}})) \in \mathcal{E}^{\approx}[\![\texttt{Unit}]\!]\rho$

PROOF: By Rule 6S from Dreyer et al.'s Figure 6.

12. We have that

$$(\mathbf{x_1''''}, \mathbf{x_2''''}) \in \mathcal{V}^{\approx}[\![\alpha]\!](\rho, \alpha \mapsto \mathbf{R}_\alpha) = \\ (\mathbf{x_1''''}, \mathbf{x_2''''}) \in R_\alpha \equiv \mathit{False}$$

PROOF: By the definition of the value relation in Dreyer et al.'s Figure 4, by the definition of R_α in 4 and by the first axiom of Dreyer et al.'s Figure 3.

13. We now have an assumption of *False* in Θ''' by 12 and 11, so we can conclude vacuously.

14. Q.E.D.

In the paper, we also use a version of the terms $\mathbf{t_u}$ and $\mathbf{t_d}$ where we replace builtin existentials with existentials encoded using the following encoding:

$$\exists \mathbf{X}.\, \tau \overset{\mathsf{def}}{=} \forall \beta.\, (\forall \alpha.\, \tau \to \beta) \to \beta$$
$$\text{pack } \langle \tau', \mathbf{t} \rangle \text{ as } \exists \mathbf{X}.\, \tau \overset{\mathsf{def}}{=} \mathbf{\Lambda}\beta.\, \lambda \mathbf{f} : (\forall \alpha.\, \tau \to \beta).\, \mathbf{f}\ [\tau']\ \mathbf{t}$$
$$\text{unpack } \mathbf{t_1} \text{ as } \langle \alpha, \mathbf{x} \rangle \text{ in } \mathbf{t_2} \overset{\mathsf{def}}{=} \mathbf{t_1}\ [\tau_2]\ (\mathbf{\Lambda}\alpha.\, \lambda \mathbf{x} : \tau_1.\, \mathbf{t_2}) \qquad \text{where } \mathbf{t_1} : \exists \alpha.\, \tau_1 \text{ and } \mathbf{t_2} : \tau_2$$

We then get the following $\mathbf{Univ}'$, $\mathbf{t_u'}$ and $\mathbf{t_d'}$:

$$\mathbf{Univ}' \overset{\mathsf{def}}{=} \forall \gamma.\, (\forall \beta.\, (\forall \alpha.\, (\alpha \to \beta) \times (\beta \to \alpha)) \to \gamma) \to \gamma$$
$$\mathbf{t_u'} \overset{\mathsf{def}}{=} \lambda x : \mathrm{Univ}'.\, x\ \texttt{Unit}\ (\Lambda \beta.\, \lambda x' : (\forall \alpha.\, (\alpha \to \beta) \times (\beta \to \alpha)). \\ \text{let } x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = x'\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit}))$$

$$\mathbf{t}'_{\mathbf{d}} \stackrel{\mathsf{def}}{=} \lambda x : \mathrm{Univ}'.\, x\ \mathtt{Unit}\ (\Lambda\beta.\, \lambda x' : (\forall\alpha.\,(\alpha \to \beta) \times (\beta \to \alpha)).$$
$$\text{let } x'' : (\mathtt{Unit} \to \beta) \times (\beta \to \mathtt{Unit}) = x'\ \mathtt{Unit} \text{ in } (x''.2\ (x''.1\ \mathtt{unit}); \omega_{\mathtt{Unit}}))$$

1.1 Proof of Theorem 2.4 for encoded existentials

Lemma 1.4 (Terms are also equivalent with encoded existentials). $\mathbf{t}'_{\mathbf{u}} \simeq_{ctx} \mathbf{t}'_{\mathbf{d}}$

Proof. The proof works similarly to the proof of Lemma 1.3.

1. SUFFICES: $\emptyset \vdash \mathbf{t}'_{\mathbf{u}} \approx^{\mathbf{log}} \mathbf{t}'_{\mathbf{d}}$
   PROOF: By Dreyer et al.'s Theorem 4.4 in two directions.
2. SUFFICES:
$$x : \mathrm{Univ}', \beta, x' : \forall\alpha.\,(\alpha \to \beta) \times (\beta \to \alpha) \vdash$$
$$\text{let } x'' : (\mathtt{Unit} \to \beta) \times (\beta \to \mathtt{Unit}) = x'\ \mathtt{Unit} \text{ in } x''.2\ (x''.1\ \mathtt{unit}) \approx^{\log}$$
$$\text{let } x'' : (\mathtt{Unit} \to \beta) \times (\beta \to \mathtt{Unit}) = x'\ \mathtt{Unit} \text{ in } (x''.2\ (x''.1\ \mathtt{unit}); \omega_{\mathtt{Unit}})$$
$$: \mathtt{Unit}$$
   PROOF: By a number of compatibility lemmas, as one would expect to be used in the proof of Dreyer et al.'s Theorem 4.3.
3. From here on, the proof is now identical to the proof of Lemma 1.3, from step 3 in that proof onwards.
4. Q.E.D.

2 Proof of Theorem 3.3

2.1 Untyped Version

The compilation of $\mathbf{t}_{\mathbf{u}}$ to the untyped λ^{σ} with Sumii and Pierce's compiler is the following.

We split the unfoldings of $\mathtt{erase}()$ and of $\mathsf{protect}_{;}/\mathsf{confine}_{;}$ for simplicity.

$$\begin{aligned}
&\text{let } y = \\
= &\left(\begin{array}{l}\mathtt{erase}(\lambda x : \mathrm{Univ}.\, \text{unpack } x \text{ as } \langle \beta, x' \rangle \text{ in let } x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}) = x'\ \mathtt{Bool} \text{ in} \\ \ x''.2\ (x''.1\ \mathtt{unit}))\end{array}\right) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathrm{Univ}\to\mathtt{Bool}}\ y \\
&\text{let } y = \\
= &\ \lambda x.\, \mathtt{erase}(\text{unpack } x \text{ as } \langle \beta, x' \rangle \text{ in let } x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}) = x'\ \mathtt{Bool} \text{ in } x''.2\ (x''.1\ \mathtt{unit})) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathrm{Univ}\to\mathtt{Bool}}\ y \\
&\text{let } y = \\
= &\ (\lambda x.\, \text{let } x' = \mathtt{erase}(x) \text{ in } \mathtt{erase}(\text{let } x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}) = x'\ \mathtt{Bool} \text{ in } x''.2\ (x''.1\ \mathtt{unit}))) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathrm{Univ}\to\mathtt{Bool}}\ y
\end{aligned}$$

$$\begin{aligned}
&\text{let } y = \\
= \ &(\lambda x.\,\text{let } x' = x \text{ in } \mathtt{erase}(\text{let } x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}) = x'\ \mathtt{Bool} \text{ in } x''.2\ (x''.1\ \mathtt{unit}))) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
&\text{let } y = \\
= \ &(\lambda x.\,\text{let } x' = x \text{ in } \mathtt{erase}((\lambda x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}).\, x''.2\ (x''.1\ \mathtt{unit}))(x'\ \mathtt{Bool}))) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
&\text{let } y = \\
= \ &(\lambda x.\,\text{let } x' = x \text{ in } \mathtt{erase}(\lambda x'' : (\mathtt{Bool} \to \beta) \times (\beta \to \mathtt{Bool}).\, x''.2\ (x''.1\ \mathtt{unit}))\mathtt{erase}(x'\ \mathtt{Bool})) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
&\text{let } y = \\
= \ &(\lambda x.\,\text{let } x' = x \text{ in } \lambda x''.\,\mathtt{erase}(x''.2\ (x''.1\ \mathtt{unit}))\mathtt{erase}(x')\ \mathtt{erase}(\mathtt{Bool})) \text{ in} \\
&\mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
= \ &\text{let } y = (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l}\lambda x''.\,\mathtt{erase}(x''.2)\mathtt{erase}((x''.1\ \mathtt{unit})) \\ (x'\ \mathtt{unit})\end{array}) \text{ in } \mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
= \ &\text{let } y = (\lambda x.\,\text{let } x' = x \text{ in } \lambda x''.\,(x''.2\ (\mathtt{erase}(x''.1)\mathtt{erase}(\mathtt{unit})))(x'\ \mathtt{unit})) \text{ in } \mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y \\
= \ &\left\{\begin{array}{l}\text{let } y = (\lambda x.\,\text{let } x' = x \text{ in } (\lambda x''.\,(x''.2\ (x''.1\ \mathtt{unit})))(x'\ \mathtt{unit})) \\ \quad \text{in } \mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ y\end{array}\right.
\end{aligned}$$

We unfold $\mathsf{protect}_{;}$.

$$\begin{aligned}
&\mathsf{protect}_{\emptyset;\mathtt{Univ}\to\mathtt{Bool}}\ \mathsf{y} \\
&\equiv \mathsf{protect}_{\emptyset;\exists\mathbf{Y}.\forall\mathbf{X}.(\mathbf{X}\to\mathbf{Y})\times(\mathbf{Y}\to\mathbf{X})}\ \mathsf{y} \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y}\ (\mathsf{confine}_{\emptyset;\exists\mathbf{Y}.\forall\mathbf{X}.(\mathbf{X}\to\mathbf{Y})\times(\mathbf{Y}\to\mathbf{X})}\ \mathsf{w}) \text{ in } \mathsf{protect}_{\emptyset;\mathtt{Bool}}\ \mathsf{a} \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y}\ (\mathsf{confine}_{\emptyset;\exists\mathbf{Y}.\forall\mathbf{X}.(\mathbf{X}\to\mathbf{Y})\times(\mathbf{Y}\to\mathbf{X})}\ \mathsf{w}) \text{ in } \mathsf{a} \\
&\eta = \mathbf{Y} \mapsto (\lambda \mathsf{y}.\,\mathsf{y}, \lambda \mathsf{y}.\,\mathsf{y}) \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y}\ (\mathsf{confine}_{\eta;\forall\mathbf{X}.(\mathbf{X}\to\mathbf{Y})\times(\mathbf{Y}\to\mathbf{X})}\ \mathsf{w}) \text{ in } \mathsf{a} \\
&\eta' = \mathbf{Y} \mapsto (\lambda \mathsf{y}.\,\mathsf{y}, \lambda \mathsf{y}.\,\mathsf{y}), \mathbf{X} \mapsto (\mathsf{seal_s}, \mathsf{unseal_s}) \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y}\ ((\lambda\_.\,\nu \mathsf{s}.\text{let } \mathsf{x}' = \mathsf{w}\ \mathtt{unit} \text{ in } \mathsf{confine}_{\eta';(\mathbf{X}\to\mathbf{Y})\times(\mathbf{Y}\to\mathbf{X})}\ \mathsf{x}')) \text{ in } \mathsf{a} \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y} \left(\begin{array}{l}\lambda\_.\,\nu s.\text{let } x' = w\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \mathsf{confine}_{\eta';(\mathbf{X}\to\mathbf{Y})}\ x1, \mathsf{confine}_{\eta';(\mathbf{Y}\to\mathbf{X})}\ x2\rangle\end{array}\right) \text{ in } \mathsf{a} \\
&= \lambda \mathsf{w}.\,\text{let } \mathsf{a} = \mathsf{y} \left(\begin{array}{l}\lambda\_.\,\nu s.\text{let } x' = w\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\,\text{let } z1 = x1\ (\mathsf{protect}_{\eta;\mathbf{X}}\ y1) \text{ in } \mathsf{confine}_{\eta;\mathbf{Y}}\ z1 \\ \ \ , \mathsf{confine}_{\eta';(\mathbf{Y}\to\mathbf{X})}\ x2\rangle\end{array}\right) \text{ in } \mathsf{a}
\end{aligned}$$

$$= \lambda \mathsf{w}.\, \text{let } \mathsf{a} = \mathsf{y} \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ (\mathsf{protect}_{\eta;\mathbf{X}}\ y1) \text{ in } \mathsf{confine}_{\eta;\mathbf{Y}}\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ (\mathsf{protect}_{\eta;\mathbf{Y}}\ y2) \text{ in } \mathsf{confine}_{\eta;\mathbf{X}}\ z2 \rangle \end{array} \right) \text{ in } \mathsf{a}$$

$$\eta' = \mathbf{Y} \mapsto (\lambda \mathsf{y}.\, \mathsf{y}, \lambda \mathsf{y}.\, \mathsf{y}), \mathbf{X} \mapsto (\mathsf{seal_s}, \mathsf{unseal_s})$$

$$= \lambda \mathsf{w}.\, \text{let } \mathsf{a} = \mathsf{y} \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ (seal_s\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } unseal_s\ z2 \rangle \end{array} \right) \text{ in } \mathsf{a}$$

$$= \lambda \mathsf{w}.\, \text{let } \mathsf{a} = \mathsf{y} \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2 \rangle \end{array} \right) \text{ in } \mathsf{a}$$

and thus the compiled $\mathbf{t_u}$ is:

$$= \left\{ \begin{array}{l} \text{let } y = (\lambda x.\, \text{let } x' = x \text{ in } (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit})))(x'\ \mathtt{unit})) \\ \quad \text{in } \lambda w.\, \text{let } a = y \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2 \rangle \end{array} \right) \text{ in } a \end{array} \right.$$

2.1.1 Proof (reductions)

$$\left\{ \begin{array}{l} \text{let } y = (\lambda x.\, \text{let } x' = x \text{ in } (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit})))(x'\ \mathtt{unit})) \\ \quad \text{in } \lambda w.\, \text{let } a = y \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2 \rangle \end{array} \right) \text{ i} \\ (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x \rangle) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \lambda w.\, \text{let } a = \\ (\lambda x.\, \text{let } x' = x \text{ in } (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit})))(x'\ \mathtt{unit})) \\ \left(\begin{array}{l} \lambda\_.\, \nu s. \text{let } x' = w \ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2 \rangle \end{array} \right) \text{ in } a \\ (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x \rangle) \end{array} \right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\text{let } x' = \left(\begin{array}{l} \lambda\_.\, \nu s.\text{let } x' = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in} \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit})))(x'\ \mathtt{unit})) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit})))(\\ \left(\begin{array}{l} \lambda\_.\, \nu s.\text{let } x' = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \mathtt{unit}) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\ \left(\begin{array}{l} \nu s.\text{let } x' = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_s)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_s = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\ \left(\begin{array}{l} \text{let } x' = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit} \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\ \left(\begin{array}{l} \text{let } x' = (\langle \lambda x.\, x, \lambda x.\, x\rangle) \text{ in} \\ \text{let } x1 = x'.1 \text{ in let } x2 = x'.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\ \left(\begin{array}{l} \text{let } x1 = \langle \lambda x.\, x, \lambda x.\, x\rangle.1 \text{ in let } x2 = \langle \lambda x.\, x, \lambda x.\, x\rangle.2 \text{ in} \\ \ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \ \ \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ \left(\begin{array}{l} \text{let } x1 = \lambda x.\, x \text{ in let } x2 = \langle \lambda x.\, x, \lambda x.\, x\rangle .2 \text{ in} \\ \langle \lambda y1.\, \text{let } z1 = x1\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ \left(\begin{array}{l} \text{let } x2 = \langle \lambda x.\, x, \lambda x.\, x\rangle .2 \text{ in} \\ \langle \lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = x2\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ \left(\begin{array}{l} \langle \lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right) \text{ in } a \end{array}\right.$$

$$(*) \hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ (\left(\begin{array}{l} \langle \lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right).2 \\ (\left(\begin{array}{l} \langle \lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right).1\ \texttt{unit})) \\ \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\left(\begin{array}{l} \langle \lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1 \\ \quad \lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2\rangle \end{array}\right).1\ \texttt{unit})) \\ \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ ((\lambda y1.\, \text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ y1) \text{ in } (\lambda p1.\, p1)\ z1)\ \texttt{unit})) \\ \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ \texttt{unit}) \text{ in } (\lambda p1.\, p1)\ z1)) \\ \text{ in } a \end{array}\right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\text{let } z1 = \lambda x.\, x\ ((\lambda q1.\, \{q1\}_\sigma)\ \texttt{unit}) \text{ in } (\lambda p1.\, p1)\ z1)) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\text{let } z1 = \lambda x.\, x\ (\{\texttt{unit}\}_\sigma) \text{ in } (\lambda p1.\, p1)\ z1)) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\text{let } z1 = \{\texttt{unit}\}_\sigma \text{ in } (\lambda p1.\, p1)\ z1)) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ ((\lambda p1.\, p1)\ \{\texttt{unit}\}_\sigma)) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda y2.\, \text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ y2) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ (\{\texttt{unit}\}_\sigma)) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ (\text{let } z2 = \lambda x.\, x\ ((\lambda p2.\, p2)\ \{\texttt{unit}\}_\sigma) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ (\text{let } z2 = \lambda x.\, x\ (\{\texttt{unit}\}_\sigma) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ (\text{let } z2 = (\{\texttt{unit}\}_\sigma) \text{ in } (\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ z2) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ ((\lambda q2.\, \text{let } \{a2\}_\sigma = q2 \text{ in } a2 \text{ else } \mathsf{wrong})\ \{\texttt{unit}\}_\sigma) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \text{let } a = \\ (\text{let } \{a2\}_\sigma = \{\texttt{unit}\}_\sigma \text{ in } a2 \text{ else } \mathsf{wrong}) \\ \ \text{in } a \end{array} \right.$$

$$\hookrightarrow \left\{ \text{let } a = \texttt{unit} \text{ in } a \right.$$

$\hookrightarrow$unit

The only difference in the reduction of `erase`($\mathbf{t_d}$) is that instead of the term $\lambda x''.\, x''.2\ (x''.1\ \texttt{unit})$ there is $(\lambda x''.\,(\lambda\_.\,\omega)\ (x''.2\ (x''.1\ \texttt{unit})))$ (recall that the ; is encoded as a dummy lambda). The reductions will proceed the same, except that after an argument is supplied for x'' (reduction marked with *), the last line will be ;omega in a instead of just in a Thus, in the end that reduces as follows:

$$\begin{aligned}
&\ldots\\
&\hookrightarrow\{\ \text{let } a = \texttt{unit};\omega \text{ in } a\\
&\hookrightarrow\{\ \text{let } a = \omega \text{ in } a\\
&\hookrightarrow\{\ \text{let } a = \omega \text{ in } a\\
&\Uparrow
\end{aligned}$$

2.2 Typed Version

We provide unfolding of definitions of Sumii and Pierce's compiler using their notation for the typed compiler. The paper's `erase`() function is $\mathcal{E}$ and protect/confine are $\mathcal{G}^+$ / $\mathcal{G}^-$ respectively. Sumii and Pierce rely on an additional function $\mathcal{C}$ to insert sealing/unsealing in this variant Pierce and Sumii [2000] which is also reported here.

Please note that this notation only affects the unfolding of internal definitions of the compiler, the actual proof does not depend on it.

Note that we compile `Unit` and the empty record {} to `unit`.

$$[\![\mathbf{t_u} : \text{Univ} \to \texttt{Unit}]\!]^{\lambda^{\mathbf{F}}}_{\lambda^{\sigma}}$$

$$= \begin{cases} \text{let } y \\ \quad = \mathcal{E}\left(\begin{array}{l}\lambda x : \text{Univ}.\, \text{unpack } x \text{ as } \langle \beta, x' \rangle \text{ in} \\ \text{let } x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = x'\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit})\end{array}\right) \\ \quad \text{in } \mathcal{G}^+(y, \text{Univ} \to \texttt{Unit}) \end{cases}$$

We split the derivation of the compilation of $\mathcal{E}$ and of $\mathcal{G}$ for simplicity.

$$\begin{aligned}
&\mathcal{E}\left(\begin{array}{l}\lambda x : \text{Univ}.\, \text{unpack } x \text{ as } \langle \beta, x' \rangle \text{ in} \\ \text{let } x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = x'\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit})\end{array}\right)\\
&= \{\lambda \mathsf{x}.\, \mathcal{E}(\text{unpack } \mathsf{x} \text{ as } \langle \beta, \mathsf{x}' \rangle \text{ in let } \mathsf{x}'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = \mathsf{x}'\ \texttt{Unit} \text{ in } \mathsf{x}''.2\ (\mathsf{x}''.1\ \texttt{unit}))\\
&= \{\ (\lambda x.\, \text{let } x' = \mathcal{E}(x) \text{ in } \mathcal{E}(\text{let } x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = x'\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit})))\\
&= \{\ (\lambda x.\, \text{let } x' = x \text{ in } \mathcal{E}(\text{let } x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}) = x'\ \texttt{Unit} \text{ in } x''.2\ (x''.1\ \texttt{unit})))
\end{aligned}$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \mathcal{E}((\lambda x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}).\, x''.2\ (x''.1\ \texttt{unit}))(x'\ \texttt{Unit}))) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \mathcal{E}(\lambda x'' : (\texttt{Unit} \to \beta) \times (\beta \to \texttt{Unit}).\, x''.2\ (x''.1\ \texttt{unit}))\ \mathcal{E}(x'\ \texttt{Unit})) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,\mathcal{E}(x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \mathcal{E}(x')\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \mathcal{E}(\texttt{Unit})\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,\mathcal{E}(x''.2)\ \mathcal{E}((x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \mathcal{E}(x')\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \mathcal{E}(\texttt{Unit})\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,(x''.2\ (\mathcal{E}(x''.1)\ \mathcal{E}(\texttt{unit})))) \\ (\text{let } x''' = \mathcal{E}(x')\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \mathcal{E}(\texttt{Unit})\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \mathcal{E}(x')\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \mathcal{E}(\texttt{Unit})\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = x'\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \mathcal{E}(\texttt{Unit})\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = x'\ \texttt{unit} \text{ in } \mathcal{C}^{-}_{\alpha}(x''', \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\alpha \to \beta) \times (\beta \to \alpha))) \end{array}) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \left(\begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \langle \mathcal{C}^{-}_{\alpha}(w1, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\alpha \to \beta)), \mathcal{C}^{-}_{\alpha}(w2, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\beta \to \alpha))\rangle) \end{array} \right)) \right.$$

$$= \left\{ (\lambda x.\,\text{let } x' = x \text{ in } \left(\begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \langle \lambda a.\,\text{let } r = (w1(\mathcal{C}^{+}_{\alpha}(a, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\alpha)))) \text{ in } \mathcal{C}^{-}_{\alpha}(r, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\beta)), \\ \lambda a.\,\text{let } r = (w2(\mathcal{C}^{+}_{\alpha}(a, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\beta)))) \text{ in } \mathcal{C}^{-}_{\alpha}(r, \langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle, (\alpha))\rangle) \end{array} \right)) \right.$$

$$= \left\{ \left(\lambda x.\,\text{let } x' = x \text{ in } \left(\begin{array}{l} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \ \langle \lambda a1.\,\text{let } r1 = w1\ \{a1\}_{\langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle} \text{ in } r1, \\ \quad \lambda a2.\,\text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle\, \texttt{Unit}\, \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle) \end{array} \right) \right) \right.$$

Below is the derivation of $\mathcal{G}$:

$$\mathcal{G}^{+}(\mathsf{y}, \text{Univ} \to \texttt{Unit})$$

$$= \left\{ \lambda w.\,\text{let } a = \mathcal{G}^{-}(w, \text{Univ}) \text{ in let } r = y\ a \text{ in } \mathcal{G}^{+}(r, \texttt{Unit}) \right.$$

$$= \left\{ \lambda w.\,\text{let } a = \mathcal{G}^{-}(w, \forall \alpha.\,(\alpha \to \beta) \times (\beta \to \alpha)) \text{ in let } r = y\ a \text{ in } r \right.$$

$$= \left\{ \begin{array}{l} \lambda w.\,\text{let } a = \\ \quad \lambda\_.\,\text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in} \\ \quad\ \text{let } c = \mathcal{C}^{-}_{\alpha}(d, k, (\alpha \to \beta) \times (\beta \to \alpha)) \text{ in } \mathcal{G}^{-}(c, (\alpha \to \beta) \times (\beta \to \alpha)) \\ \quad \text{in let } r = y\ a \text{ in } r \end{array} \right.$$

$$
= \left\{ \begin{array}{l}
\lambda w.\, \text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\ \text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in } \langle \mathcal{C}^-_\alpha(d1, k, (\alpha \to \beta)), \mathcal{C}^-_\alpha(d2, k, (\beta \to \alpha)) \rangle \\
\ \ \text{in let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in } \langle \mathcal{G}^-(c1, (\alpha \to \beta)), \mathcal{G}^-(c2, (\beta \to \alpha)) \rangle
\end{array} \right) \\
\quad \text{in let } r = y\ a \text{ in } r
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
\lambda w.\, \text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\quad \left(\begin{array}{l}
\text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\
\ \langle \lambda u1.\, \text{let } p1 = d1\ \mathcal{C}^- + \alpha(u1, k, \alpha) \text{ in } \mathcal{C}^-_\alpha(p1, k, \beta), \\
\ \ \lambda u2.\, \text{let } p2 = d2\ \mathcal{C}^- + \alpha(u2, k, \beta) \text{ in } \mathcal{C}^-_\alpha(p2, k, \alpha) \rangle
\end{array} \right) \\
\quad \text{in } \left(\begin{array}{l}
\text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\
\ \langle \lambda t1.\, \text{let } e1 = \mathcal{G}^+(t1, \alpha) \text{ in let } s1 = c1\ e1 \text{ in } \mathcal{G}^-(s1, \beta), \\
\ \ \lambda t2.\, \text{let } e2 = \mathcal{G}^+(t2, \beta) \text{ in let } s2 = c2\ e2 \text{ in } \mathcal{G}^-(s2, \alpha) \rangle
\end{array} \right)
\end{array} \right) \\
\quad \text{in let } r = y\ a \text{ in } r
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
\lambda w.\, \text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\quad \left(\begin{array}{l}
\text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\
\ \langle \lambda u1.\, \text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\
\ \ \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle
\end{array} \right) \\
\quad \text{in } \left(\begin{array}{l}
\text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\
\ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\
\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle
\end{array} \right)
\end{array} \right) \\
\quad \text{in let } r = y\ a \text{ in } r
\end{array} \right.
$$

Term $\mathbf{t_u}$ is thus compiled into:

$$
\left\{ \begin{array}{l}
\text{let } y = \left(\lambda x.\, \text{let } x' = x \text{ in } \left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\
\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\
\ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } r1, \\
\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle)
\end{array} \right) \right) \\
\text{in } \lambda w.\, \text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\quad \left(\begin{array}{l}
\text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\
\ \langle \lambda u1.\, \text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\
\ \ \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle
\end{array} \right) \\
\quad \text{in } \left(\begin{array}{l}
\text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\
\ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\
\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle
\end{array} \right)
\end{array} \right) \\
\qquad \text{in let } r = y\ a \text{ in } r
\end{array} \right.
$$

As we encode $\mathsf{t}; \mathsf{t}'$ as $(\lambda\_.\, \mathsf{t}')\ \mathsf{t}$, term $\mathbf{t_d}$ is compiled into:

. . .

$$\left\{\begin{array}{l}
\text{let } y = \left(\lambda x.\, \text{let } x' = x \text{ in } \left(\begin{array}{l}
(\lambda x''.\, (\lambda\_.\, \omega)(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\
\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\
\ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \\
\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle)
\end{array}\right)\right) \\
\text{in } \lambda w.\, \text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\ \left(\begin{array}{l}
\text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\
\ \langle \lambda u1.\, \text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\
\ \ \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle
\end{array}\right) \\
\ \text{in } \left(\begin{array}{l}
\text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\
\ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\
\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle
\end{array}\right)
\end{array}\right) \\
\qquad \text{in let } r = y\ a \text{ in } r
\end{array}\right.$$

2.2.1 Proof (reductions)

$$\left\{\begin{array}{l}
\left(\begin{array}{l}
\text{let } y = \left(\lambda x.\, \text{let } x' = x \text{ in } \left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\
\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\
\ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \\
\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle)
\end{array}\right)\right) \\
\text{in } \lambda w. \begin{array}{l}
\text{let } a = \left(\begin{array}{l}
\lambda\_.\, \text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\
\ \left(\begin{array}{l}
\text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\
\ \langle \lambda u1.\, \text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\
\ \ \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle
\end{array}\right) \\
\ \text{in } \left(\begin{array}{l}
\text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\
\ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\
\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle
\end{array}\right)
\end{array}\right) \\
\text{in let } r = y\ a \text{ in } r
\end{array}
\end{array}\right) \\
(\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x \rangle)
\end{array}\right.$$

$$\hookrightarrow \left\{ \begin{array}{l} \left(\lambda w.\ \begin{array}{l} \text{let } a = \begin{pmatrix} \lambda\_.\,\text{let } k = \nu s.s \text{ in let } d = w\ \texttt{unit} \text{ in let } c = \\ \begin{pmatrix} \text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\ \langle \lambda u1.\,\text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\ \ \lambda u2.\,\text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong}\rangle \end{pmatrix} \\ \text{in } \begin{pmatrix} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \langle \lambda t1.\,\text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \ \lambda t2.\,\text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle \end{pmatrix} \end{pmatrix} \\ \text{in } \text{let } r = \begin{pmatrix} \lambda x.\,\text{let } x' = x \text{ in } \begin{pmatrix} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \langle \lambda a1.\,\text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \\ \ \lambda a2.\,\text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle) \end{pmatrix} \end{pmatrix} \\ \quad a \text{ in } r \end{array} \right) \\ (\lambda\_.\,\langle \lambda x.\,x, \lambda x.\,x\rangle) \end{array} \right.$$

$$\hookrightarrow \left\{ \left(\begin{array}{l} \text{let } a = \begin{pmatrix} \lambda\_.\,\text{let } k = \nu s.s \text{ in let } d = (\lambda\_.\,\langle \lambda x.\,x, \lambda x.\,x\rangle)\ \texttt{unit} \text{ in let } c = \\ \begin{pmatrix} \text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\ \langle \lambda u1.\,\text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\ \ \lambda u2.\,\text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong}\rangle \end{pmatrix} \\ \text{in } \begin{pmatrix} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \langle \lambda t1.\,\text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \ \lambda t2.\,\text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle \end{pmatrix} \end{pmatrix} \\ \text{in } \text{let } r = \begin{pmatrix} \lambda x.\,\text{let } x' = x \text{ in } \begin{pmatrix} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \langle \lambda a1.\,\text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \\ \ \lambda a2.\,\text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle) \end{pmatrix} \end{pmatrix} \\ \quad a \text{ in } r \end{array} \right) \right.$$

$$\hookrightarrow \left\{ \left(\begin{array}{l} \text{let } r = \begin{pmatrix} \lambda x.\,\text{let } x' = x \text{ in } \begin{pmatrix} (\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\ (\text{let } x''' = x'\ \texttt{unit} \text{ in} \\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \langle \lambda a1.\,\text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \\ \ \lambda a2.\,\text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle) \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} \lambda\_.\,\text{let } k = \nu s.s \text{ in let } d = (\lambda\_.\,\langle \lambda x.\,x, \lambda x.\,x\rangle)\ \texttt{unit} \text{ in let } c = \\ \begin{pmatrix} \text{let } d1 = d.1 \text{ in let } d2 = d.2 \text{ in} \\ \langle \lambda u1.\,\text{let } p1 = d1\ \{u1\}_k \text{ in } p1, \\ \ \lambda u2.\,\text{let } p2 = d2\ u2 \text{ in let } \{j\}_k = p2 \text{ in } j \text{ else } \mathsf{wrong}\rangle \end{pmatrix} \\ \text{in } \begin{pmatrix} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \langle \lambda t1.\,\text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \ \lambda t2.\,\text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle \end{pmatrix} \end{pmatrix} \text{ in } r \end{array} \right) \right.$$

$$\equiv \left\{\begin{array}{l}
(\lambda r.\, r) \\
\left(\begin{array}{l}
\left(\lambda x.\, \mathrm{let}\ x' = x\ \mathrm{in} \left(\begin{array}{l}
(\lambda x''.\,(x''.2\ (x''.1\ \mathtt{unit})))\ (\mathrm{let}\ x''' = x'\ \mathtt{unit}\ \mathrm{in} \\
\ \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \\
\quad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle)
\end{array}\right)\right) \\
\left(\begin{array}{l}
\lambda\_.\, \mathrm{let}\ k = \nu s.s\ \mathrm{in}\ \mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\ \langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_k\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_k = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array}\right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\ \langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array}\right)
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l}
(\lambda r.\, r) \\
\left(\begin{array}{l}
\mathrm{let}\ x' = \left(\begin{array}{l}
\lambda\_.\, \mathrm{let}\ k = \nu s.s\ \mathrm{in}\ \mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\ \langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_k\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_k = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array}\right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\ \langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array}\right)
\end{array}\right) \\
\mathrm{in} \left(\begin{array}{l}
(\lambda x''.\,(x''.2\ (x''.1\ \mathtt{unit})))\ (\mathrm{let}\ x''' = x'\ \mathtt{unit}\ \mathrm{in} \\
\ \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \\
\quad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle)
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

$$\hookrightarrow \left\{\begin{array}{l}
(\lambda r.\, r) \\
\left(\begin{array}{l}
\left(\begin{array}{l}
(\lambda x''.\,(x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\lambda\_.\, \mathrm{let}\ k = \nu s.s\ \mathrm{in}\ \mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\ \langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_k\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_k = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array}\right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\ \langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array}\right)
\end{array}\right)\ \mathtt{unit} \\
\quad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad\qquad)\\
\qquad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array}\right)
\end{array}\right)
\end{array}\right.$$

$$\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ k = \nu s.s\ \mathrm{in}\ \mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_k\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_k = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in}\ \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\quad\quad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\quad \mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad\qquad) \\
\quad\quad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ k = \sigma\ \mathrm{in}\ \mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_k\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_k = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in}\ \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\quad\quad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\quad \mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad\qquad) \\
\quad\quad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ d = (\lambda\_.\, \langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathtt{unit}\ \mathrm{in}\ \mathrm{let}\ c = \\
\quad \left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_\sigma\ \mathrm{in}\ p1, \\
\quad \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in}\ \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\quad \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\quad\quad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\quad \mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad\qquad) \\
\quad\quad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.$$

$$
\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ d = (\langle \lambda x.\, x, \lambda x.\, x\rangle)\ \mathrm{in}\ \mathrm{let}\ c = \\
\left(\begin{array}{l}
\mathrm{let}\ d1 = d.1\ \mathrm{in}\ \mathrm{let}\ d2 = d.2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_\sigma\ \mathrm{in}\ p1, \\
\ \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\ \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\qquad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad) \\
\qquad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.
$$

$$
\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ c = \\
\left(\begin{array}{l}
\mathrm{let}\ d1 = (\langle \lambda x.\, x, \lambda x.\, x\rangle).1\ \mathrm{in}\ \mathrm{let}\ d2 = (\langle \lambda x.\, x, \lambda x.\, x\rangle).2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_\sigma\ \mathrm{in}\ p1, \\
\ \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\ \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\qquad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad) \\
\qquad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.
$$

$$
\hookrightarrow \left\{ \begin{array}{l}
(\lambda r.\, r) \\
\left(\left(\begin{array}{l}
(\lambda x''.\, (x''.2\ (x''.1\ \mathtt{unit}))) \\
(\mathrm{let}\ x''' = \left(\begin{array}{l}
\mathrm{let}\ c = \\
\left(\begin{array}{l}
\mathrm{let}\ d1 = \lambda x.\, x\ \mathrm{in}\ \mathrm{let}\ d2 = (\langle \lambda x.\, x, \lambda x.\, x\rangle).2\ \mathrm{in} \\
\langle \lambda u1.\, \mathrm{let}\ p1 = d1\ \{u1\}_\sigma\ \mathrm{in}\ p1, \\
\ \lambda u2.\, \mathrm{let}\ p2 = d2\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \\
\quad \mathrm{in} \left(\begin{array}{l}
\mathrm{let}\ c1 = c.1\ \mathrm{in}\ \mathrm{let}\ c2 = c.2\ \mathrm{in} \\
\langle \lambda t1.\, \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = c1\ e1\ \mathrm{in}\ s1, \\
\ \lambda t2.\, \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = c2\ e2\ \mathrm{in}\ s2\rangle
\end{array} \right)
\end{array} \right) \\
\qquad \mathrm{let}\ w1 = x'''.1\ \mathrm{in}\ \mathrm{let}\ w2 = x'''.2\ \mathrm{in} \\
\mathrm{in}\ \ \langle \lambda a1.\, \mathrm{let}\ r1 = w1\ \{a1\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1, \qquad) \\
\qquad \lambda a2.\, \mathrm{let}\ r2 = w2\ a2\ \mathrm{in}\ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong}\rangle
\end{array} \right) \right)
\end{array} \right.
$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\left(\begin{array}{l} (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \left(\begin{array}{l} \text{let } c = \\ \quad \left(\begin{array}{l} \text{let } d2 = (\langle \lambda x.\, x, \lambda x.\, x \rangle).2 \text{ in} \\ \ \langle \lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1, \\ \quad \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle \end{array} \right) \\ \text{in} \left(\begin{array}{l} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \quad \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle \end{array} \right) \end{array} \right) \\ \qquad \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \text{in} \quad \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad) \\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle \end{array} \right) \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\left(\begin{array}{l} (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \left(\begin{array}{l} \text{let } c = \\ \quad \left(\begin{array}{l} \text{let } d2 = \lambda x.\, x \text{ in} \\ \ \langle \lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1, \\ \quad \lambda u2.\, \text{let } p2 = d2\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle \end{array} \right) \\ \text{in} \left(\begin{array}{l} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \quad \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle \end{array} \right) \end{array} \right) \\ \qquad \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \text{in} \quad \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad) \\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle \end{array} \right) \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\left(\begin{array}{l} (\lambda x''.\, (x''.2\ (x''.1\ \texttt{unit}))) \\ (\text{let } x''' = \left(\begin{array}{l} \text{let } c = \\ \quad \left(\begin{array}{l} \langle \lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1, \\ \lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \rangle \end{array} \right) \\ \text{in} \left(\begin{array}{l} \text{let } c1 = c.1 \text{ in let } c2 = c.2 \text{ in} \\ \ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1, \\ \quad \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2 \rangle \end{array} \right) \end{array} \right) \\ \qquad \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in} \\ \text{in} \quad \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad) \\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \rangle \end{array} \right) \right) \end{array} \right.$$

$$\hookrightarrow\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ (\text{let } x''' = \left(\left(\begin{array}{l}\text{let } c1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\\ \text{in let } c2 = \left(\begin{array}{l}\langle \lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1,\\ \lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong}\rangle\end{array}\right).2 \text{ in}\\ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = c1\ e1 \text{ in } s1,\\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle\end{array}\right.\right.\right)\\ \quad\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in}\\ \text{in}\ \ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad\qquad\qquad)\\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle\end{array}\right.\right.\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ (\text{let } x''' = \left(\left(\begin{array}{l}\text{let } c2 = \left(\begin{array}{l}\langle \lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1,\\ \lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong}\rangle\end{array}\right).2 \text{ in}\\ \ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1,\\ \ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle\end{array}\right)\right)\\ \quad\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in}\\ \text{in}\ \ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad\qquad\qquad)\\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle\end{array}\right.\right)\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ (\text{let } x''' = \left(\left(\begin{array}{l}\text{let } c2 = \lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \text{ in}\\ \ \langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1,\\ \ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = c2\ e2 \text{ in } s2\rangle\end{array}\right)\right)\\ \quad\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in}\\ \text{in}\ \ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad\qquad\qquad)\\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle\end{array}\right.\right)\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ (\text{let } x''' = \left(\begin{array}{l}\langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1,\\ \ \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong})\end{array}\right.\\ \quad\ \text{let } w1 = x'''.1 \text{ in let } w2 = x'''.2 \text{ in}\\ \text{in}\ \ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} \text{ in } r1, \qquad\qquad\qquad)\\ \qquad \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\!\langle \texttt{Unit} \rangle\!\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong}\rangle\end{array}\right.\right.\end{array}\right.$$

$$\hookrightarrow\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ \text{let } w1 = (\lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1)\\ \text{in let } w2 = \left(\begin{array}{l}\langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1,\\ \quad \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else w}\end{array}\right.\\ \langle \lambda a1.\, \text{let } r1 = w1\ \{a1\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } r1,\\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \textsf{wrong}\rangle\end{array}\right.\right.\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ \text{let } w2 = \left(\begin{array}{l}\langle \lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1,\\ \quad \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \textsf{wrong})\end{array}\right.\\ \langle \lambda a1.\, \text{let } r1 = (\lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1)\ \{a1\}_{\langle\langle \texttt{Unit} \rangle}\\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \textsf{wrong}\rangle\end{array}\right.\right.\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\left(\begin{array}{l}(\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ \text{let } w2 = \lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 = (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \textsf{wrong})\ e2 \text{ i}\\ \langle \lambda a1.\, \text{let } r1 = (\lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 = (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1)\ \{a1\}_{\langle\langle \texttt{Unit} \rangle}\\ \ \lambda a2.\, \text{let } r2 = w2\ a2 \text{ in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \textsf{wrong}\rangle\end{array}\right.\right.\end{array}\right.$$

$$\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\begin{array}{l}\lambda x''.\,(x''.2\ (x''.1\ \texttt{unit})))\\ \langle\begin{array}{l}\lambda a1.\, \text{let } r1 = (\lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 =\\ (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1)\ \{a1\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } r1\end{array}\\ ,\ \begin{array}{l}\lambda a2.\, \text{let } r2 = (\lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 =\\ (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \textsf{wrong})\ e2 \text{ in } s2)\ a2 \text{ in }\rangle\\ \text{let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \textsf{wrong}\end{array}\end{array}\right)\end{array}\right.$$

$$\hookrightarrow\hookrightarrow\hookrightarrow\left\{\begin{array}{l}(\lambda r.\, r)\\ \left(\begin{array}{l}\left(\begin{array}{l}\lambda a2.\, \text{let } r2 = (\lambda t2.\, \text{let } e2 = t2 \text{ in let } s2 =\\ (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \textsf{wrong})\ e2 \text{ in } s2)\ a2 \text{ in}\\ \text{let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \textsf{wrong}\end{array}\right)\\ \left(\left(\begin{array}{l}\lambda a1.\, \text{let } r1 = (\lambda t1.\, \text{let } e1 = t1 \text{ in let } s1 =\\ (\lambda u1.\, \text{let } p1 = (\lambda x.\, x)\ \{u1\}_\sigma \text{ in } p1)\ e1 \text{ in } s1)\ \{a1\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } r1\end{array}\right)\ \texttt{unit}\right)\end{array}\right)\end{array}\right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \left(\left(\begin{array}{l} \mathrm{let}\ r1 = (\lambda t1.\ \mathrm{let}\ e1 = t1\ \mathrm{in}\ \mathrm{let}\ s1 = \\ \ (\lambda u1.\ \mathrm{let}\ p1 = (\lambda x.\, x)\ \{u1\}_\sigma\ \mathrm{in}\ p1)\ e1\ \mathrm{in}\ s1)\ \{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ r1 \end{array} \right) \right) \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \left(\left(\begin{array}{l} \mathrm{let}\ r1 = \mathrm{let}\ e1 = \{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ \mathrm{let}\ s1 = \\ \ (\lambda u1.\ \mathrm{let}\ p1 = (\lambda x.\, x)\ \{u1\}_\sigma\ \mathrm{in}\ p1)\ e1\ \mathrm{in}\ s1\ \mathrm{in}\ r1 \end{array} \right) \right) \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \left(\left(\begin{array}{l} \mathrm{let}\ r1 = \mathrm{let}\ s1 = \\ \ (\lambda u1.\ \mathrm{let}\ p1 = (\lambda x.\, x)\ \{u1\}_\sigma\ \mathrm{in}\ p1)\ \{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\ \mathrm{in}\ s1\ \mathrm{in}\ r1 \end{array} \right) \right) \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \left(\left(\begin{array}{l} \mathrm{let}\ r1 = \mathrm{let}\ s1 = \\ \ (\mathrm{let}\ p1 = (\lambda x.\, x)\ \{\{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\}_\sigma\ \mathrm{in}\ p1)\ \mathrm{in}\ s1\ \mathrm{in}\ r1 \end{array} \right) \right) \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \left(\left(\begin{array}{l} \mathrm{let}\ r1 = \mathrm{let}\ s1 = \\ \ (\mathrm{let}\ p1 = (\{\{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\}_\sigma\ \mathrm{in}\ p1)\ \mathrm{in}\ s1\ \mathrm{in}\ r1 \end{array} \right) \right) \end{array} \right) \end{array} \right.$$

$$\hookrightarrow\hookrightarrow\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \left(\begin{array}{l} \lambda a2.\ \mathrm{let}\ r2 = (\lambda t2.\ \mathrm{let}\ e2 = t2\ \mathrm{in}\ \mathrm{let}\ s2 = \\ \ (\lambda u2.\ \mathrm{let}\ p2 = (\lambda x.\, x)\ u2\ \mathrm{in}\ \mathrm{let}\ \{j\}_\sigma = p2\ \mathrm{in}\ j\ \mathrm{else}\ \mathsf{wrong})\ e2\ \mathrm{in}\ s2)\ a2\ \mathrm{in} \\ \ \ \mathrm{let}\ \{x6\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle} = r2\ \mathrm{in}\ x6\ \mathrm{else}\ \mathsf{wrong} \end{array} \right) \\ \{\{\mathtt{unit}\}_{\langle\!\langle \mathtt{Unit} \rangle\!\rangle}\}_\sigma \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \begin{array}{l} \text{let } e2 = \{\{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle}\}_\sigma \\ \text{in let } s2 = (\lambda u2.\, \text{let } p2 = (\lambda x.\, x)\ u2 \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong})\ e2 \text{ in } s2 \end{array} \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \text{let } s2 = \text{let } p2 = (\lambda x.\, x)\ \{\{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle}\}_\sigma \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \text{ in } s2 \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \text{let } s2 = \text{let } p2 = \{\{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle}\}_\sigma \text{ in let } \{j\}_\sigma = p2 \text{ in } j \text{ else } \mathsf{wrong} \text{ in } s2 \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \text{let } s2 = \text{let } \{j\}_\sigma = \{\{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle}\}_\sigma \text{ in } j \text{ else } \mathsf{wrong} \text{ in } s2 \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \text{let } s2 = \{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } s2 \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ \left(\begin{array}{l} \text{let } r2 = \{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle} \\ \text{in let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = r2 \text{ in } x6 \text{ else } \mathsf{wrong} \end{array} \right) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ (\text{let } \{x6\}_{\langle\langle \texttt{Unit} \rangle\rangle} = \{\texttt{unit}\}_{\langle\langle \texttt{Unit} \rangle\rangle} \text{ in } x6 \text{ else } \mathsf{wrong}) \end{array} \right.$$

$$\hookrightarrow \left\{ \begin{array}{l} (\lambda r.\, r) \\ (\texttt{unit}) \end{array} \right.$$

$$\hookrightarrow \left\{ \texttt{unit} \right.$$

3 Proof (and code) for Theorem 4.1

The following are the literal encodings of $\mathfrak{C}^B[\lfloor \mathbf{t_u} \rfloor]$ and $\mathfrak{C}^B[\lfloor \mathbf{t_d} \rfloor]$ for use in the interpreter provided by Ahmed et al. [2017].

This is available at http://www.ccs.neu.edu/home/dijamner/paramblame/artifact/. These terms were constructed and tested with some kind assistance by Jeremy Siek and others, which we thank them for.

As discussed in the paper, the fact that we have a non-value-polymorphic source language $\lambda^{\mathbf{F}}$ and a value-polymorphic target language λ^{B}, means that we need to introduce a form of thunking for polymorphic functions: the type $\forall \mathbf{X}.\, \tau$ is mapped to $\lfloor \forall \mathbf{X}.\, \tau \rfloor \overset{\mathsf{def}}{=} \forall X.\, \texttt{Unit} \to \lfloor \tau \rfloor$ and type abstractions $\mathbf{\Lambda X.\, t}$ are mapped to $\lfloor \mathbf{\Lambda X.\,t} \rfloor \overset{\text{def}}{=} \Lambda X.\,\lambda\_.\,\lfloor \mathbf{t} \rfloor$. However, the artifact by Ahmed et al. does not support a unit type, so in the terms below, we replace type Unit by type `int` and we use value `3` in place of the value `unit`.

```
let tu : (exists B. forall A. int -> < A -> B, B -> A >) -> bool =
 lam(x:(exists B. forall A. int -> < A -> B, B -> A >)).
 unpack[forall A. int -> < A -> B, B -> A >, bool] B, x1 = x in
  let x2 : < bool -> B,B -> bool > = x1 [bool] 3 in
  (snd x2) ((fst x2) true)
in let vuniv : exists B. forall A. int -> < A -> B, B -> A > =
 (pack ∗, Lam A. lam (z:int). < lam (x:A). x:A => ∗, lam (x:∗). x:∗ => A >
  in B. forall A. int -> < A -> B, B -> A >)
in tu vuniv
```

```
let omega : bool ->∗ =
 lam(x:bool).
  (lam(f :∗ ->∗). (f (f :∗ ->∗ =>∗ )))
  ((lam(f :∗ ->∗). (f (f :∗ ->∗ =>∗ ))) : (∗->∗)->∗ =>∗ ->∗)
in
let td : (exists B. forall A. int -> < A-> B, B ->A >) -> bool =
 lam(x:(exists B. forall A. int -> <A -> B, B -> A>)).
 unpack[forall A. int -> <A -> B, B -> A>, bool] B,x1 = x in
  let x2 : < bool -> B, B -> bool > = x1 [bool] 3 in
   let z : bool = (snd x2) ((fst x2) true) in
    let _ :∗ = omega true in z
in let vuniv : exists B. forall A. int -> < A -> B, B -> A> =
 (pack ∗, Lam A. lam (z:int). <lam (x:A). x:A => ∗, lam (x:∗). x:∗ => A>
  in B. forall A. int -> < A -> B, B -> A >)
in td vuniv
```

References

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