# On the Semantic Expressiveness of Recursive Types – Recap


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What is the relative semantic expressiveness of **iso-** and *equi*-recursive types?

- open question
- clarifies the design of emerging languages
- better understanding of how to answer language expressiveness questions

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- Rely on Fully-Abstract Compilation to
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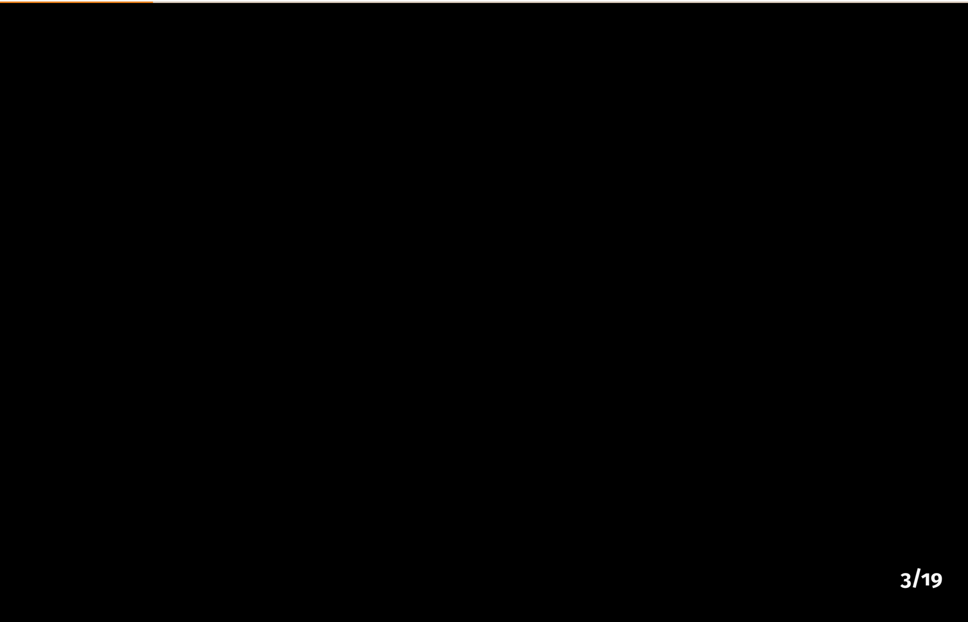
# How?

- Rely on Fully-Abstract Compilation to
  - phrase relative semantic expressiveness
  - compare language expressiveness
- Prove FAC between $\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}$ and $\lambda^{\mu}_{E}$ (and between them and $\lambda^{\mathsf{fx}}$) using approximate Backtranslations and step-indexed logical approximations (or, "directional" step-indexed logical relations)

# Our Contributions, Visually



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$$\boldsymbol{\lambda}^{\mu}_{\mathbf{I}} \underset{\langle\!\langle \cdot \rangle\!\rangle^{\lambda^{\mu}_{E}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}\, n}}{\overset{[\![ \cdot ]\!]^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mu}_{E}}}{\rightleftarrows}} \lambda^{\mu}_{E}$$

$[\![ \cdot ]\!]^{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}}_{\lambda^{\mu}_{E}}$ erases $\mathbf{fold}$ / $\mathbf{unfold}$, $\langle\!\langle \cdot \rangle\!\rangle^{\lambda^{\mu}_{E}}_{\boldsymbol{\lambda}^{\mu}_{\mathbf{I}}\, n}$ is approximate

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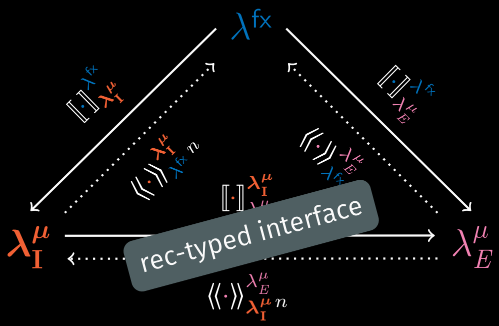


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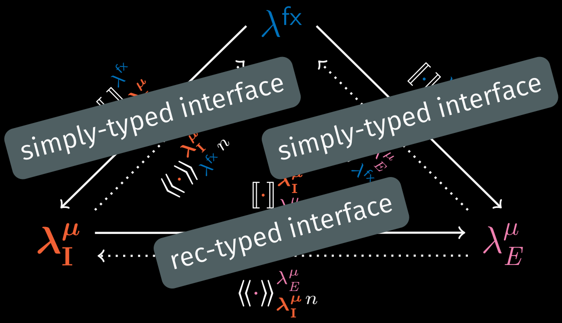


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- use prog. eq: lets us compare language abstractions (i.e., hiding)

  e.g., $\mathbf{pack}\,\langle \mathrm{N}, \mathbf{0}\rangle\,\mathbf{as}\ \tau$ and $\mathbf{pack}\,\langle \mathrm{N}, \mathbf{1}\rangle\,\mathbf{as}\ \tau$

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- behaviour: $\Uparrow$ vs $\Downarrow$
  alternatives exist (e.g., traces) but this is operational-semantics-based

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- $[\![\cdot]\!]$: identity and erase $\mathbf{fold}$ / $\mathbf{unfold}$
  $\boldsymbol{\lambda}^{\boldsymbol{\mu}}_{\mathbf{I}}$ & $\lambda^{\mu}_{E}$ semantics are identical (almost)

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$$\begin{gathered}(\forall \mathbf{C}.\mathbf{C}[\mathbf{t_1}]\Downarrow \iff \mathbf{C}[\mathbf{t_2}]\Downarrow) \\ \Updownarrow \\ (\forall C.C[[\![\mathbf{t_1}]\!]]\Downarrow \iff C[[\![\mathbf{t_2}]\!]]\Downarrow)\end{gathered}$$

# Preservation via Step-Idx Log. Approx.

$$\mathbf{t}_1 \simeq_{\mathrm{ctx}} \mathbf{t}_2$$

ctx. eq. preservation

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(1)

(3)

$\mathbf{t_2} \lesssim_{\_} [\![\mathbf{t_2}]\!]$

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# Overapproximation

Define a cross-language logical approximation $(\mathcal{V}\llbracket\cdot\rrbracket_{\triangledown}, \mathcal{E}\llbracket\cdot\rrbracket_{\triangledown}, \cdots)$ $\triangledown$ is the direction

- $\mathbf{t} \lesssim_n \llbracket\mathbf{t}\rrbracket$ : by def. $\cdots$ $\mathbf{t}$ and $\llbracket\mathbf{t}\rrbracket$ are in the obs.:

$$O(W)_{\lesssim} \stackrel{\text{def}}{=} \left\{ (\mathbf{t}, \llbracket\mathbf{t}\rrbracket) \;\middle|\; \begin{array}{l} \text{if } lev(W) > n \text{ and } \mathbf{t} \hookrightarrow^{\mathbf{n}} \mathbf{v} \\ \text{then } \exists \mathbf{k}.\ \llbracket\mathbf{t}\rrbracket \hookrightarrow^{k} v \end{array} \right\}$$

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- $\mathbf{t} \gtrsim_n [\![\mathbf{t}]\!]$ :
  same, flipped implication

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- $\langle\!\langle\cdot\rangle\!\rangle_n : C \to \mathbf{C}$
  approximate $C$'s coinductive derivation
- sufficient because FAC cares about co-termination

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$$\mathbf{BtT}_{\mathbf{0};\tau} \stackrel{\text{def}}{=} \mathbf{Unit}$$

$$\mathbf{BtT}_{\mathbf{n+1};\tau} \stackrel{\text{def}}{=} \begin{cases} \mathbf{Unit} \uplus \mathbf{Unit} & \text{if } \tau = \mathit{Unit} \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \to \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \to \tau' \\ (\mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}} \uplus \mathbf{BtT}_{\mathbf{n};\boldsymbol{\tau}'}) \uplus \mathbf{Unit} & \text{if } \tau = \tau \uplus \tau' \\ \mathbf{BtT}_{\mathbf{n+1};\boldsymbol{\tau}'[\mu\alpha.\boldsymbol{\tau}'/\alpha]} \uplus \mathbf{Unit} & \text{if } \tau = \mu\alpha.\,\tau' \end{cases}$$

# Backtranslation Example and Relation

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$$\mathcal{V}\,[\![\mathbf{Unit}]\!]_{\triangledown} \stackrel{\text{def}}{=} \{(\mathbf{unit}, unit)\}$$

Need a special value relation:

$$\mathcal{V}\,[\![\mathbf{EmulT}_{\mathbf{n+1};\tau}]\!]_{\triangledown} \stackrel{\text{def}}{=} \Big\{(\mathbf{v}, v) \mid \text{either } \mathbf{v} = \mathbf{inr\ unit}$$
$$\text{or } \tau = Unit \text{ and } \exists \mathbf{v}'.\ \mathbf{v} = \mathbf{inl\ v}' \text{ and}$$
$$(\mathbf{v}', v) \in \mathcal{V}\,[\![\mathbf{Unit}]\!]_{\triangledown}\Big\}$$

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  which needs upgrading and downgrading (for well-typedness)
- inject/extract: to fix typing of context interface

# Technicality #1: Emulate

- translate to ‘the same’ term
  (switch to TR, p 146)

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- typing error! need to lose a step!

# Technicality #1.1: Upgrade/Downgrade

- needed to ensure well-typedness only

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- needed to ensure well-typedness only
- recursively traverse a term and add or lose a level (i.e., an $\mathbf{inl}$ )
  (switch to TR, only in blue though, p 96)

# Technicality #1.1: Upgrade/Downgrade

$$\text{if } (n < m \text{ and } p = \mathtt{precise}) \text{ or } (\triangledown = \lesssim \text{ and } p = \mathtt{imprecise})$$
$$\Gamma \vdash \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m+d;p;}\tau}$$
$$\text{then } \Gamma \vdash \mathbf{downgrade}_{\mathbf{m;d;}\tau}\ \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m;p;}\tau}$$

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$$\Gamma \vdash \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m;p;}\tau}$$
$$\text{then } \Gamma \vdash \mathbf{upgrade}_{\mathbf{m;d;}\tau}\ \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m+d;p;}\tau}$$

# Technicality #1: Emulate

if $(m > n$ and $p = \texttt{precise})$ or $(\triangledown = \lesssim$ and $p = \texttt{imprecise})$

$\Gamma \vdash t : \tau$

then $\texttt{toEmul}_{\mathbf{m;p}}(\Gamma) \vdash \mathbf{emulate_m}(\Gamma \vdash t : \tau) \triangledown_n t : \mathbf{EmulT}_{\mathbf{m;p};\tau}$

Key:

if $\tau \triangleq \sigma$

and $\texttt{ftv}(\tau) = \texttt{ftv}(\sigma) = \varnothing$

then $\mathbf{BtT}_{\mathbf{n};\tau} = \mathbf{BtT}_{\mathbf{n};\sigma}$ for all $n$

# Technicality #2: Inject/Extract

- Since $\mathtt{t} : \boldsymbol{\tau}$ implies $[\![\mathtt{t}]\!] : \overbrace{[\![\boldsymbol{\tau}]\!]}^{\tau}$
- And $C\,[: \tau]$

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- $\langle\!\langle C \rangle\!\rangle_{\mathbf{n}}\,[: ?]$

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- $\langle\!\langle C \rangle\!\rangle_{\mathbf{n}} \left[: \mathbf{BtT}_{\mathbf{n};\tau}\right]$
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- $\langle\!\langle C \rangle\!\rangle_{\mathbf{n}}\,\left[\cdot : \mathbf{BtT}_{\mathbf{n};\tau}\right]$
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- $\langle\!\langle [\cdot] \rangle\!\rangle_{\mathbf{n}} = \left[\mathbf{inject}_{\mathbf{n};\boldsymbol{\tau}}\ \cdot\right]$
  $\mathbf{inject}_{\mathbf{n};\boldsymbol{\tau}} : \boldsymbol{\tau} \rightarrow \mathbf{BtT}_{\mathbf{n};[\![\boldsymbol{\tau}]\!]}$

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# Technicality #2: Inject/Extract

If $(m \geq n$ and $p = \texttt{precise})$ or $(\triangledown = \lesssim$ and $p = \texttt{imprecise})$

then if $\Gamma \vdash \mathbf{t} \triangledown_n t : \boldsymbol{\tau}$

then $\Gamma \vdash \mathbf{inject}_{\mathbf{m};\boldsymbol{\tau}}\ \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m};\mathbf{p};\texttt{isToEq}(\boldsymbol{\tau})}$

if $\Gamma \vdash \mathbf{t} \triangledown_n t : \mathbf{EmulT}_{\mathbf{m};\mathbf{p};\texttt{isToEq}(\boldsymbol{\tau})}$

then $\Gamma \vdash \mathbf{extract}_{\mathbf{m};\boldsymbol{\tau}}\ \mathbf{t} \triangledown_n t : \boldsymbol{\tau}$